

# ON HOMEOMORPHISM TYPE OF SYMMETRIC PRODUCTS OF COMPACT RIEMANN SURFACES WITH PUNCTURES

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## Abstract

Let  $M_{g,k}^2$  and  $M_{g',k'}^2$  be compact Riemann surfaces with punctures ( $g, g' \geq 0$  — genuses,  $k, k' \geq 1$  — number of punctures). For any Hausdorff space  $X$  the quotient space  $\text{Sym}^n X := X^n/S_n$  is the  $n$ -th symmetric product of  $X$ ,  $n \geq 2$ . It is well known, that  $\text{Sym}^n M_{g,k}^2$  is a smooth quasi-projective variety. Open manifolds  $\text{Sym}^n M_{g,k}^2$  and  $\text{Sym}^n M_{g',k'}^2$  are homotopy equivalent iff  $2g + k = 2g' + k'$ .

**Blagojević-Grujić-Živaljević Conjecture (2003).** *Fix any  $n \geq 2$ , and two pairs  $(g, k)$  and  $(g', k')$  with the condition  $2g + k = 2g' + k'$ . If  $g \neq g'$ , then open manifolds  $\text{Sym}^n M_{g,k}^2$  and  $\text{Sym}^n M_{g',k'}^2$  are not continuously homeomorphic.*

The conjecture was proved in the paper [1] by P. Blagojević, V. Grujić and R. Živaljević for the case  $\max(g, g') \geq \frac{n}{2}$  (this implies the case  $n = 2$ ). As far as the author knows, up to this moment there were no results if  $\max(g, g') < \frac{n}{2}$ .

The aim of this paper is to prove the conjecture in full generality.

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*To my son Alesha*

## 1 Introduction

Let  $X$  be a Hausdorff space. The quotient space  $\text{Sym}^n X := X^n/S_n$  is called the  $n$ -th symmetric product of  $X$  for any integer  $n \geq 2$ . It is easy to see that the functor  $\text{Sym}^n$  is a homotopy functor. If  $X$  is a finite (countable) simplicial complex, then spaces  $\text{Sym}^n X$ ,  $n \geq 2$ , admit at least two natural structures of a finite (countable) simplicial complexes. Therefore, if  $X$  is homotopy equivalent to a countable (finite) CW-complex, then spaces  $\text{Sym}^n X$ ,  $n \geq 2$ , are also homotopy equivalent to countable (finite) CW-complexes.

The (co)homology of  $\text{Sym}^n X$  was investigated by Nakaoka, Dold, Thom, Mattuck, Macdonald, Milgram and many other mathematicians starting from 1950-s. The main disadvantage of this functor is the following

**Fact  $\alpha$ .** *Suppose  $M^m$  is any topological  $m$ -dimensional manifold, where  $m \geq 3$ . Then for any  $n \geq 2$  the space  $\text{Sym}^n M^m$  is not even a homology manifold.*

This fact is an easy consequence of the

**Fact  $\beta$ .** *For  $m \geq 3$ , the space  $\text{Sym}^2 \mathbb{R}^m$  is homeomorphic to  $\mathbb{R}^m \times \text{ConeInt}(\mathbb{R}P^{m-1})$ . Here by  $\text{ConeInt}(X)$  we denote the open cone  $\text{Cone}(X) \setminus X$ .*

On the other hand, for 2-dimensional manifolds there symmetric products are always manifolds. This is the consequence of the

**Fact  $\gamma$ .** *There exists a canonical homeomorphism  $\text{Sym}^n \mathbb{C} \cong \mathbb{C}^n$  for all  $n \geq 2$ . ( $n$  roots of a unital polynomial  $\leftrightarrow$  its  $n$  coefficients)*

This fact also implies that if we have an arbitrary Riemann surface  $M^2$  (compact or noncompact), then the topological manifold  $\text{Sym}^n M^2$ ,  $n \geq 2$ , inherits some natural structure of a complex  $n$ -dimensional manifold. Therefore, in this case we have a natural  $C^\omega$  and  $C^\infty$  structures on the manifold  $\text{Sym}^n M^2$  (these are the weakening of a holomorphic structure).

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But, if we have two Riemann surfaces  $M^2$  and  $N^2$ , that are only  $C^\infty$ -diffeomorphic, then smooth manifolds  $\text{Sym}^n M^2$  and  $\text{Sym}^n N^2$  *a priori* could not be  $C^\infty$ -diffeomorphic. We should recall the fundamental fact that 2-dimensional  $C^\infty$ -manifolds (closed or open) are  $C^\infty$ -diffeomorphic *iff* they are just homeomorphic. Thus, we would like to pose the following

**Conjecture 1.** *Let  $M^2$  and  $N^2$  be Riemann surfaces, that are closed or closed with a finite number of punctures. Suppose that  $M^2$  and  $N^2$  are homeomorphic ( $= C^\infty$ -diffeomorphic). Then smooth manifolds  $\text{Sym}^n M^2$  and  $\text{Sym}^n N^2$  are  $C^\infty$ -diffeomorphic for all  $n \geq 2$ .*

The following fact is the main theorem of [9].

**Fact  $\delta$ .** *The space  $\text{Sym}^n S^1$ ,  $n \geq 2$ , is homeomorphic to the  $D^{n-1}$ -bundle over  $S^1$ , which is trivial for odd  $n$ , and non-oriented for even  $n$ .*

We also need the following main result of [11].

**Fact  $\varepsilon$ .** *The space  $\text{Sym}^n \bigvee_1^m S^1$ ,  $n \geq 2, m \geq 1$ , is homotopy equivalent to  $\text{Sk}^n T^m$ . Here, the cell structure on the torus  $T^m$  is the direct product of the minimal cell structure on  $S^1$  (this structure has one 0-cell).*

The following fact is a folklore.

**Fact  $\zeta$ .** *Suppose  $X$  is a connected CW-complex. Then  $\pi_1(\text{Sym}^n X) = \pi_1(X)^{ab} = H_1(X; \mathbb{Z})$  for all  $n \geq 2$ .*

Let us now focus on symmetric products of compact Riemann surfaces with a finite number of punctures. We denote by  $M_{g,k}^2$  a Riemann surface of genus  $g \geq 0$ , and  $k$  distinct points removed,  $k \geq 1$ .

It is obvious that  $M_{g,k}^2$  is homotopy equivalent to the bouquet  $\bigvee_1^s S^1$ , where  $s = 2g + k - 1$ . The fact  $\zeta$  implies that  $\pi_1(\text{Sym}^n M_{g,k}^2) = \mathbb{Z}^{2g+k-1}$  for all  $n \geq 2$ . Let us fix any  $n \geq 2$ , and two pairs  $(g, k)$  and  $(g', k')$ . From above, we have that open manifolds  $\text{Sym}^n M_{g,k}^2$  and  $\text{Sym}^n M_{g',k'}^2$  are homotopy equivalent *iff*  $2g + k = 2g' + k'$ .

Now we are ready to formulate the

**Blagojević-Grujić-Živaljević Conjecture (2003).** *Fix any  $n \geq 2$ , and two pairs  $(g, k)$  and  $(g', k')$  with the condition  $2g + k = 2g' + k'$ . If  $g \neq g'$ , then open manifolds  $\text{Sym}^n M_{g,k}^2$  and  $\text{Sym}^n M_{g',k'}^2$  are not continuously homeomorphic.*

This conjecture was posed by Rade Živaljević, Pavle Blagojević and Vladimir Grujić in [1], at R. Živaljević lecture on the conference 18th British Topology Meeting, Manchester, September 2003, and also in [2]. According to Živaljević (personal communication), they were very much inspired and influenced by the work of Kostadin Trenčevski and Dončo Dimovski [12] and [13], see in particular the conjecture after Theorem A.2 in [12]. The conjecture was proved in [1] for the case  $\max(g, g') \geq \frac{n}{2}$  (this implies the case  $n = 2$ ). As far as the author knows, up to this moment there were no results if  $\max(g, g') < \frac{n}{2}$ .

The aim of this paper is to prove the following generalization of this conjecture.

**Theorem 1.** *Fix any  $n \geq 2$ , and two pairs  $(g, k)$  and  $(g', k')$  with the condition  $2g + k = 2g' + k'$ . If  $g \neq g'$ , then open manifolds  $\text{Sym}^n M_{g,k}^2 \times \mathbb{R}^N$  and  $\text{Sym}^n M_{g',k'}^2 \times \mathbb{R}^N$  are not continuously homeomorphic for all  $N \geq 0$ .*

**Here is the plan of the proof of Theorem 1.**

Set  $s := 2g + k - 1 = 2g' + k' - 1$ . If  $s = 0$  or  $1$ , then  $g = g'$ . So, we have  $s \geq 2$ . Fix a pair  $(g, k)$  with the condition  $2g + k - 1 = s$ . We will denote by  $\mathbb{Z}_2$  the field  $\mathbb{Z}/2\mathbb{Z}$ .

**Step 1.** The space  $\text{Sym}^n M_{g,k}^2 \sim \text{Sk}^n T^s$  has torsionless integral homology. Therefore, we have the ring isomorphism  $H^*(\text{Sym}^n M_{g,k}^2; \mathbb{Z}_2) \cong H^*(\text{Sym}^n M_{g,k}^2; \mathbb{Z}) \otimes \mathbb{Z}_2$ .

**Step 2.** The integral cohomology ring  $H^*(\text{Sym}^n M_{g,k}^2; \mathbb{Z})$  is equal to the cutted exterior algebra  $\Lambda_{\mathbb{Z}}^{\leq n}(\alpha_1, \alpha_2, \dots, \alpha_s)$  for some  $\mathbb{Z}$ -basis  $\alpha_1, \alpha_2, \dots, \alpha_s$  of  $H^1(\text{Sym}^n M_{g,k}^2; \mathbb{Z})$ . Thus, the ring  $H^*(\text{Sym}^n M_{g,k}^2; \mathbb{Z}_2)$  is equal to  $\Lambda_{\mathbb{Z}_2}^{\leq n}(\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_s)$  for some (any)  $\mathbb{Z}_2$ -basis  $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_s$  of  $H^1(\text{Sym}^n M_{g,k}^2; \mathbb{Z}_2)$ .

**Step 3.** The open manifold  $\text{Sym}^n M_{g,k}^2$  is a Zariski-open subset of the smooth projective variety  $\text{Sym}^n M_g^2$ , where  $M_g^2$  is the initial compact Riemann surface without punctures. The total Chern class of the complex manifold  $\text{Sym}^n M_g^2$  was computed by Macdonald in his famous paper [7]. The inclusion

$i_{(n)}: \text{Sym}^n M_{g,k}^2 \rightarrow \text{Sym}^n M_g^2$  induce the ring homomorphism  $i_{(n)}^*: H^*(\text{Sym}^n M_g^2; \mathbb{Z}) \rightarrow H^*(\text{Sym}^n M_{g,k}^2; \mathbb{Z})$ , and  $i_{(n)}^*(c_1(\text{Sym}^n M_g^2)) = c_1(\text{Sym}^n M_{g,k}^2)$ .

From Macdonald's calculations one can easily derive that

$$c_1(\text{Sym}^n M_{g,k}^2) = -(\alpha_1 \smile \alpha_2 + \alpha_3 \smile \alpha_4 + \dots + \alpha_{2g-1} \smile \alpha_{2g})$$

for some  $\mathbb{Z}$ -basis  $\alpha_1, \alpha_2, \dots, \alpha_s$  of  $H^1(\text{Sym}^n M_{g,k}^2; \mathbb{Z})$ .

**Step 4.** Suppose we have a complex vector bundle  $\xi: E \rightarrow B$  with the fiber  $\mathbb{C}^n$  and the base  $B$ , which is a connected ENR (compact, or non-compact and homotopy equivalent to a finite polyhedron). Then the Stiefel-Whitney classes  $w_k$  of the realization  $\xi_{\mathbb{R}}: E_{\mathbb{R}} \rightarrow B$  can be computed from the Chern classes  $c_l$  of the initial vector bundle  $\xi: E \rightarrow B$  as follows:

$$w_{2k+1}(\xi_{\mathbb{R}}) = 0 \quad \forall k \geq 0; \quad w_{2k}(\xi_{\mathbb{R}}) = \rho_2(c_k(\xi)) \quad \forall k \geq 1.$$

Here,  $\rho_2: H^*(B; \mathbb{Z}) \rightarrow H^*(B; \mathbb{Z}_2)$  is the reduction homomorphism.

The statement of this step is a well known fact.

**Step 5.** Combining two previous steps, we have that

$$w_2(\text{Sym}^n M_{g,k}^2) = \bar{\alpha}_1 \smile \bar{\alpha}_2 + \bar{\alpha}_3 \smile \bar{\alpha}_4 + \dots + \bar{\alpha}_{2g-1} \smile \bar{\alpha}_{2g}$$

for some  $\mathbb{Z}_2$ -basis  $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_s$  of  $H^1(\text{Sym}^n M_{g,k}^2; \mathbb{Z}_2)$ . As one has

$$H^2(\text{Sym}^n M_{g,k}^2; \mathbb{Z}_2) \cong \Lambda^2(H^1(\text{Sym}^n M_{g,k}^2; \mathbb{Z}_2)),$$

we get

$$w_2(\text{Sym}^n M_{g,k}^2) = \bar{\alpha}_1 \wedge \bar{\alpha}_2 + \bar{\alpha}_3 \wedge \bar{\alpha}_4 + \dots + \bar{\alpha}_{2g-1} \wedge \bar{\alpha}_{2g}.$$

#### Step 6. (Topological invariance of Stiefel-Whitney classes for open smooth manifolds)

Suppose we have closed smooth connected manifolds  $M^n$  and  $N^n$ . By celebrated Wu formula, if  $f: M^n \rightarrow N^n$  is a homotopy equivalence, then  $f^*(w_k(N^n)) = w_k(M^n)$  for all  $k \geq 1$ . It is the famous *Homotopy invariance* of Stiefel-Whitney classes for closed manifolds.

But, the trivial example  $M^2 = S^1 \times \mathbb{R}^1$  and  $N^2 =$  (open Möbius strip) shows us that even  $w_1$  is not a *homotopy* invariant for open manifolds.

Now we want to pose the following

**Conjecture 2.** Suppose we have a purely continuous homeomorphism  $f: M^n \rightarrow N^n$  of two open connected smooth manifolds, which are homotopy equivalent to a finite polyhedron. Then  $f^*(w_k(N^n)) = w_k(M^n)$  for all  $1 \leq k \leq n$ .

*Remark.* This conjecture is trivially true for  $w_1$  (a loop preserve or change the orientation), and for  $w_n = 0$ .

Below we will prove this conjecture for  $w_2$  with the following additional condition: the abelian groups  $H_1(M^n; \mathbb{Z})$  and  $H_2(M^n; \mathbb{Z})$  are torsionless and  $H_2(M^n; \mathbb{Z})$  is generated by the images of continuous mappings of torus  $T^2$  to  $M^n$ .

**Step 7.** Combining the steps 5 and 6, we get that the topological type of the open manifold  $\text{Sym}^n M_{g,k}^2$  determines the genus  $g$ . Moreover, as Stiefel-Whitney classes are invariant under taking the direct product with the euclidian spaces  $\mathbb{R}^N$ ,  $N \geq 0$ , we conclude the proof of Theorem 1.

## 2 Steps 1-2

Suppose  $Z$  is a finite connected CW-complex. Fix any  $n \geq 2$  and the commutative ring  $R$ . Denote by  $X$  the  $n$ -skeleton  $\text{Sk}^n(Z)$ . It is evident that the inclusion  $i: X \hookrightarrow Z$  induce the isomorphism  $i^*: H^k(Z; R) \cong H^k(X; R)$  for all  $0 \leq k \leq n-1$ . Suppose also that the algebraic boundary  $\partial\sigma$  (with  $\mathbb{Z}$  coefficients) of any  $(n+1)$ -dimensional cell  $\sigma$  of  $Z$  is equal to zero. Then, it is easy to see that  $i^*: H^n(Z; R) \rightarrow H^n(X; R)$  is also an isomorphism.

As the induced mapping  $i^*: H^*(Z; R) \rightarrow H^*(X; R)$  is a ring homomorphism and  $\dim X \leq n$ , we get that the ring  $H^*(X; R)$  is just the  $(n+1)$ -cutted ring  $H^{*\leq n}(Z; R)$ . Moreover, the mapping  $i^*: H^*(Z; R) \rightarrow H^*(X; R)$  just cuts the  $* \geq (n+1)$  part of  $H^*(Z; R)$ .

All the above requirements are satisfied for  $Z = T^s$  (with standard minimal cell structure) and any  $n \geq 2$ . Therefore, we have proved the steps 1-2.

### 3 Step 3

Let  $M_g^2$  be an arbitrary compact Riemann surface of genus  $g$  and without punctures. It is well known that the ring  $H^*(M_g^2; \mathbb{Z})$  is torsionless. Also one can choose the  $\mathbb{Z}$ -basis  $\gamma_1, \gamma_2, \dots, \gamma_{2g} \in H^1(M_g^2; \mathbb{Z})$  with the property

$$\gamma_i \gamma_j = 0 \text{ unless } i - j = \pm g; \quad \gamma_i \gamma_{i+g} = -\gamma_{i+g} \gamma_i = \delta = [M_g^2] \quad (1 \leq i \leq g).$$

Macdonald in his famous paper [7] proved that the ring  $H^*(\text{Sym}^n M_g^2; \mathbb{Z})$  is torsionless and gave an explicit description of this ring. But, Macdonald's proof contained several gaps. The full verification of Macdonald's theorem was made by the author in preprint [5] using only algebraic topology tools. Another verification, which uses heavily algebraic geometry, was made in 2002 by del Baño [3].

For  $g = 0$ , one has  $M_0^2 = \mathbb{CP}^1$  and  $\text{Sym}^n \mathbb{CP}^1 = \mathbb{CP}^n$ . Also for any  $k \geq 1$ , we have  $M_{0,k}^2 = \mathbb{C} \setminus \{\mu_1, \dots, \mu_{k-1}\}$  and  $\text{Sym}^n M_{0,k}^2$  is an open domain in  $\mathbb{C}^n$ . Therefore, the open manifold  $\text{Sym}^n M_{0,k}^2$  is parallizable. So, suppose  $g \geq 1$ .

One has the canonical projection  $\pi_n: (M_g^2)^n \rightarrow \text{Sym}^n M_g^2$ , which induces the isomorphism

$$\pi_n^*: H^*(\text{Sym}^n M_g^2; \mathbb{Q}) \cong H^*((M_g^2)^n; \mathbb{Q})^{S_n}.$$

Macdonald's theorem tells that the torsionless ring

$$H^*(\text{Sym}^n M_g^2; \mathbb{Z}) \subset H^*((M_g^2)^n; \mathbb{Z})^{S_n} = (H^*(M_g^2; \mathbb{Z})^{\otimes n})^{S_n}$$

has multiplicative generators

$$\xi_1 := \chi(\gamma_1), \xi_2 := \chi(\gamma_2), \dots, \xi_g := \chi(\gamma_g); \quad \xi'_1 := \chi(\gamma_{g+1}), \xi'_2 := \chi(\gamma_{g+2}), \dots, \xi'_g := \chi(\gamma_{2g}) \text{ and } \eta := \chi(\delta),$$

where

$$\chi(\omega) := \omega \otimes 1 \otimes \dots \otimes 1 + 1 \otimes \omega \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes \omega \quad \text{for all } \omega \in H^*(M_g^2; \mathbb{Z}).$$

Macdonald also computed the total Chern class of the complex manifold  $\text{Sym}^n M_g^2$  (see [7], theorem (14.5)):

$$c(\text{Sym}^n M_g^2) = (1 + \eta)^{n-2g+1} (1 + \eta - \xi_1 \xi'_1) (1 + \eta - \xi_2 \xi'_2) \dots (1 + \eta - \xi_g \xi'_g).$$

The open manifold  $\text{Sym}^n M_{g,k}^2$  is obviously a Zariski-open subset of the smooth projective variety  $\text{Sym}^n M_g^2$ .

The inclusion  $i_{(n)}: \text{Sym}^n M_{g,k}^2 \rightarrow \text{Sym}^n M_g^2$  induce the ring homomorphism  $i_{(n)}^*: H^*(\text{Sym}^n M_g^2; \mathbb{Z}) \rightarrow H^*(\text{Sym}^n M_{g,k}^2; \mathbb{Z})$ , and  $i_{(n)}^*(c_1(\text{Sym}^n M_g^2)) = c_1(\text{Sym}^n M_{g,k}^2)$ .

Let  $i: M_{g,k}^2 \rightarrow M_g^2$  be the inclusion mapping. Then  $i_n := (i)^n: (M_{g,k}^2)^n \rightarrow (M_g^2)^n$  and  $i_{(n)} := \text{Sym}^n i: \text{Sym}^n M_{g,k}^2 \rightarrow \text{Sym}^n M_g^2$  are the corresponding mappings.

Let us made the following auxiliary observation. Suppose  $X$  and  $Y$  are connected ENR's (compact, or non-compact and homotopy equivalent to a finite polyhedron). Suppose also that homology  $H_*(X; \mathbb{Z}), H_*(Y; \mathbb{Z})$  and  $H_*(\text{Sym}^n X; \mathbb{Z}), H_*(\text{Sym}^n Y; \mathbb{Z})$  are torsion-free for some fixed  $n \geq 2$ . Let  $i: X \rightarrow Y$  be a continuous mapping. One has the following commutative diagram:

$$\begin{array}{ccc} X \times \dots \times X & \xrightarrow{i_n} & Y \times \dots \times Y \\ \downarrow & & \downarrow \\ \text{Sym}^n X & \xrightarrow{i_{(n)}} & \text{Sym}^n Y \end{array}$$

Let us use the notation

$$\chi(\omega) := \omega \otimes 1 \otimes \dots \otimes 1 + 1 \otimes \omega \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes \omega.$$

If  $\omega \in H^*(X; \mathbb{Z})$ , then symmetric tensor  $\chi_X(\omega) \in H^*(X^n; \mathbb{Z})$  lies in the subring  $H^*(\text{Sym}^n X; \mathbb{Z})$ . (This fact follows from Nakaoka's Theorem (2.7) in [10]. Also it was rediscovered by the author (see [4], Integrality Lemma)).

For the above diagram one has  $i_{(n)}^*(\chi_Y(\omega)) = \chi_X(i^*(\omega))$  for all  $\omega \in H^*(Y; \mathbb{Z})$ . So, for the case  $X := M_{g,k}^2$  and  $Y := M_g^2$  we get

$$i_{(n)}^*(\eta) = i_{(n)}^*(\chi(\delta)) = \chi(i^*(\delta)) = \chi(0) = 0.$$

Therefore, one has the formula for the total Chern class of the manifold  $\text{Sym}^n M_{g,k}^2$ :

$$c(\text{Sym}^n M_{g,k}^2) = (1 - i_{(n)}^*(\xi_1) i_{(n)}^*(\xi'_1))(1 - i_{(n)}^*(\xi_2) i_{(n)}^*(\xi'_2)) \dots (1 - i_{(n)}^*(\xi_g) i_{(n)}^*(\xi'_g)).$$

This formula implies the presentation for the first Chern class

$$\begin{aligned} c_1(\text{Sym}^n M_{g,k}^2) &= -[i_{(n)}^*(\xi_1) i_{(n)}^*(\xi'_1) + i_{(n)}^*(\xi_2) i_{(n)}^*(\xi'_2) + \dots + i_{(n)}^*(\xi_g) i_{(n)}^*(\xi'_g)] = \\ &= -[\chi(i^*(\gamma_1))\chi(i^*(\gamma_{g+1})) + \chi(i^*(\gamma_2))\chi(i^*(\gamma_{g+2})) + \dots + \chi(i^*(\gamma_g))\chi(i^*(\gamma_{2g}))]. \end{aligned}$$

Now we need a one more observation. Let  $X$  and  $Y$  be as above. Suppose that the induces homomorphism  $i_*: \pi_1(X) \rightarrow \pi_1(Y)$  is an epimorphism. This implies that  $i_*: H_1(X; \mathbb{Z}) \rightarrow H_1(Y; \mathbb{Z})$  is also an epimorphism. As the abelian groups  $H_1(X; \mathbb{Z})$  and  $H_1(Y; \mathbb{Z})$  are torsionless, one has that  $i^*: H^1(Y; \mathbb{Z}) \rightarrow H^1(X; \mathbb{Z})$  is a monomorphism and the image  $i^*(H^1(Y; \mathbb{Z}))$  is the direct summand of  $H^1(X; \mathbb{Z})$ .

For our case  $X := M_{g,k}^2$  and  $Y := M_g^2$  the above condition is satisfied. So, we have that  $i^*(\gamma_1), i^*(\gamma_2), \dots, i^*(\gamma_{2g})$  is a part of some  $\mathbb{Z}$ -basis  $\varepsilon_1, \dots, \varepsilon_{2g}, \varepsilon_{2g+1}, \dots, \varepsilon_s$  of the free abelian group  $H^1(M_{g,k}^2; \mathbb{Z})$ .

The theorem 1 from the author's paper [5] implies the following

**Fact  $\eta$ .** *Let  $X$  be a connected ENR, compact or non-compact and homotopy equivalent to a finite polyhedron. Suppose  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s$  is a  $\mathbb{Z}$ -basis of the free abelian group  $H^1(X; \mathbb{Z})$ . Then elements  $\chi(\varepsilon_1), \dots, \chi(\varepsilon_s)$  form a  $\mathbb{Z}$ -basis of the free abelian group  $H^1(\text{Sym}^n X; \mathbb{Z})$  for all  $n \geq 2$ .*

Due to this fact and the above observations we get that

$$c_1(\text{Sym}^n M_{g,k}^2) = -(\alpha_1 \smile \alpha_2 + \alpha_3 \smile \alpha_4 + \dots + \alpha_{2g-1} \smile \alpha_{2g})$$

for some  $\mathbb{Z}$ -basis  $\alpha_1, \alpha_2, \dots, \alpha_s$  of  $H^1(\text{Sym}^n M_{g,k}^2; \mathbb{Z})$ . The Step 3 is proved.

## 4 Step 5

As the homology  $H_*(\text{Sym}^n M_{g,k}^2; \mathbb{Z})$  is torsionless, one has  $H^*(\text{Sym}^n M_{g,k}^2; \mathbb{Z}_2) = H^*(\text{Sym}^n M_{g,k}^2; \mathbb{Z}) \otimes \mathbb{Z}_2$ . Combining two previous steps, we have that

$$w_2(\text{Sym}^n M_{g,k}^2) = \bar{\alpha}_1 \smile \bar{\alpha}_2 + \bar{\alpha}_3 \smile \bar{\alpha}_4 + \dots + \bar{\alpha}_{2g-1} \smile \bar{\alpha}_{2g}$$

for some  $\mathbb{Z}_2$ -basis  $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_s$  of  $H^1(\text{Sym}^n M_{g,k}^2; \mathbb{Z}_2)$ . As one has

$$H^2(\text{Sym}^n M_{g,k}^2; \mathbb{Z}_2) \cong \Lambda^2(H^1(\text{Sym}^n M_{g,k}^2; \mathbb{Z}_2)),$$

we get

$$w_2(\text{Sym}^n M_{g,k}^2) = \bar{\alpha}_1 \wedge \bar{\alpha}_2 + \bar{\alpha}_3 \wedge \bar{\alpha}_4 + \dots + \bar{\alpha}_{2g-1} \wedge \bar{\alpha}_{2g}.$$

## 5 Step 6

Above we posed the following

**Conjecture 2.** *Suppose we have a purely continuous homeomorphism  $f: M^n \rightarrow N^n$  of two open connected smooth manifolds, which are homotopy equivalent to a finite polyhedron. Then  $f^*(w_k(N^n)) = w_k(M^n)$  for all  $1 \leq k \leq n$ .*

*Remark.* This conjecture is trivially true for  $w_1$  (a loop preserve or change the orientation), and for  $w_n = 0$ .

Now we will prove this conjecture for  $w_2$  with the following additional condition: *the abelian groups  $H_1(M^n; \mathbb{Z})$  and  $H_2(M^n; \mathbb{Z})$  are torsionless and  $H_2(M^n; \mathbb{Z})$  is generated by the images of continuous mappings of torus  $T^2$  to  $M^n$ .*

As Stiefel-Whitney classes for smooth manifolds are trivially invariant under taking the direct product with euclidian spaces  $\mathbb{R}^N$ ,  $N \geq 0$ , we can assume that the dimension  $n$  is as big as we want. Suppose that  $n \geq 6$ .

By the above condition on  $H_1(M^n; \mathbb{Z})$  and  $H_2(M^n; \mathbb{Z})$  one has that

- (1)  $H_2(M^n; \mathbb{Z}_2) = H_2(M^n; \mathbb{Z}) \otimes \mathbb{Z}_2$  and
- (2) there exists a  $\mathbb{Z}_2$ -basis  $\alpha_1 = g_{1*}[T^2], \alpha_2 = g_{2*}[T^2], \dots, \alpha_t = g_{t*}[T^2]$  in  $H_2(M^n; \mathbb{Z}_2)$  for some smooth embeddings  $g_1, \dots, g_t: T^2 \rightarrow M^n$ .

Therefore, the topological invariance of  $w_2$  in this case is a consequence of the following

**Lemma 1.** *Suppose we have a purely continuous homeomorphism  $f: M^n \rightarrow N^n$  of two open connected smooth manifolds of dimension  $n \geq 6$ , which are homotopy equivalent to a finite polyhedron. Let  $g: T^2 \rightarrow M^n$  be a smooth embedding. Then one has  $w_{2M}(g_*[T^2]) = w_{2N}((fg)_*[T^2])$ .*

**Proof.** We need the following auxiliary observation.

Suppose  $L^n$  is a connected smooth manifold, closed or open and homotopy equivalent to a compact polyhedron. Let  $K^p$  be a connected smooth closed manifold of dimension  $1 \leq p \leq n-1$ . Suppose  $g: K^p \rightarrow L^n$  is a continuous mapping, and  $g(K^p) \subset U^n$ , where  $U^n$  is an open domain in  $L^n$  of finite homotopy type. Then  $w_{pL}(g_*[K^p]) = w_{pU}(g_*[K^p])$ , where  $w_{pL}$  and  $w_{pU}$  are the  $p$ -th Stiefel-Whitney classes of manifolds  $L^n$  and  $U^n$  respectively. This fact is well known. We will call this observation a Locality property.

Now we go back to the Lemma conditions. Let  $U^n$  be a very small tubular neighbourhood of the submanifold  $g(T^2) \subset M^n$ . We take  $U^n$  with the boundary  $\partial U^n = L^{n-1}$ , which is a smooth fiber bundle over  $g(T^2)$  with the fiber  $S^{n-3}$ . One trivially has  $\pi_1(L^{n-1}) = \pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ .

We get the continuous homeomorphism  $f: U^n \rightarrow f(U^n) =: V^n \subset N^n$ , where  $\text{int}(V^n)$  is an open domain in  $N^n$ . The open manifold  $\text{int}(V^n)$  inherits a smooth structure from  $N^n$ .

From above observation we have that  $w_{2M}(g_*[T^2]) = w_{2U}(g_*[T^2])$  and  $w_{2N}((fg)_*[T^2]) = w_{2V}((fg)_*[T^2])$ . Therefore, we need to prove the equality  $w_{2U}(g_*[T^2]) = w_{2V}((fg)_*[T^2])$ . Here  $f: U^n \rightarrow V^n$  is a purely continuous homeomorphism,  $U^n$  is a compact connected smooth manifold with the boundary, and  $\text{int}(V^n)$  has some smooth structure. Moreover, the boundary  $\partial U^n = L^{n-1}$  is connected and has a free abelian fundamental group.

Let us take the doubles  $\hat{U}^n := U^n \cup_{L^{n-1}} U^n$  and  $\hat{V}^n := V^n \cup_{f(L^{n-1})} V^n$ . The manifold  $\hat{U}^n$  is smooth. The double  $\hat{V}^n$  has two open domains with equal smooth structures (left part and right part), but a priori we have no natural smooth structure around the topologically locally flat codimension 1 submanifold  $f(L^{n-1}) \subset \hat{V}^n$ .

The continuous homeomorphism  $f: U^n \rightarrow V^n$  can be naturally extended to continuous homeomorphism  $\hat{f}: \hat{U}^n \rightarrow \hat{V}^n$ .

By the above Locality property and topological (even homotopy) invariance of Stiefel-Whitney classes for closed smooth manifolds, to conclude the proof of Lemma 1 it is sufficient to have the following

**Lemma 2.** *Suppose  $V^n$  is a topological connected compact  $n$ -manifold,  $n \geq 6$ , with the connected boundary  $\partial V^n$  such that  $\pi_1(\partial V^n)$  is a free abelian group. Suppose also that  $\Sigma$  is a smooth structure on the interior  $\text{int}(V^n)$ . Take the double  $\hat{V}^n := V_+^n \cup_{\partial V^n} V_-^n$ , where  $V_\pm^n$  are the two copies of  $V^n$ . Let  $\hat{\Sigma}$  be the union of the smooth structures  $\Sigma_\pm = \Sigma$  on the union of open domains  $\text{int}(V_+^n) \cup \text{int}(V_-^n)$ . Then for any  $\varepsilon > 0$ , there exists some smooth structure  $\hat{\Sigma}_\varepsilon$  on the whole double  $\hat{V}^n$  such that the structures  $\hat{\Sigma}_\varepsilon$  and  $\hat{\Sigma}$  coincides on the open subset  $U_\varepsilon := \{x \in \hat{V}^n | d(x, \partial V^n) > \varepsilon\}$ . Here  $d(\cdot, \cdot)$  is an arbitrary fixed metric on the metrizable compact  $\hat{V}^n$ .*

**Proof.** One has a natural involution  $\tau: \hat{V}^n \rightarrow \hat{V}^n$ , which permutes the left and the right parts  $V_\pm^n$ , and  $\tau(x) = x$  iff  $x \in \partial V^n$ . Let us denote by  $L^{n-1}$  the boundary  $\partial V^n$ .

By the topological collaring theorem there exists a collar  $\overline{C}_+ \approx L^{n-1} \times [0, 1) \subset V_+^n$ . Let us consider the collar interior  $C_+$ , which is homeomorphic to  $L^{n-1} \times (0, 1)$ . The smooth structure  $\Sigma$  induce some smooth structure  $\Theta_+$  on the open domain  $C_+$ .

By the celebrated Product Structure Theorem (see [6], Essay I), there exist some smooth structure  $\Theta_0$  on  $L^{n-1}$  and a diffeomorphism  $f_+: L_{\Theta_0}^{n-1} \times (0, 1) \rightarrow (L^{n-1} \times (0, 1))_{\Theta_+}$ . By the involution  $\tau$ , we get the symmetric collar  $\overline{C}_- \subset V_-^n$ , the symmetric smooth structure  $\Theta_-$  on the open domain  $C_-$  and the symmetric diffeomorphism  $f_-: L_{\Theta_0}^{n-1} \times (-1, 0) \rightarrow (L^{n-1} \times (-1, 0))_{\Theta_-}$ .

Fix any  $\varepsilon > 0$ . There exists sufficiently small  $0 < \delta < \frac{1}{2}$  such that if  $x \in \hat{V}^n$  and  $d(x, L^{n-1}) > \varepsilon$ , then

$$x \notin f_+(L_{\Theta_0}^{n-1} \times (0, \delta]) \sqcup L^{n-1} \sqcup f_-(L_{\Theta_0}^{n-1} \times [-\delta, 0)).$$

Let us consider the subset  $f_+(L_{\Theta_0}^{n-1} \times (0, \delta]) \sqcup L^{n-1}$ . By standard argumentation it is a topological  $h$ -cobordism. But, the fundamental group  $\pi = \pi_1(L^{n-1})$  is a free abelian group, so the Whitehead torsion  $\text{Wh}(\pi)$  is zero (it is a classical theorem of Bass-Heller-Swan). Thus, the  $h$ -cobordism  $f_+(L_{\Theta_0}^{n-1} \times (0, \delta]) \sqcup L^{n-1}$  is an  $s$ -cobordism. By the  $s$ -cobordism theorem in topological category this  $s$ -cobordism is a cylinder.

Therefore, one has a homeomorphism  $h_+ : L_{\Theta_0}^{n-1} \times \{\delta\} \rightarrow L^{n-1} = \partial V^n$  and a respective homeomorphism of cylinders

$$\bar{h}_+ : f_+(L_{\Theta_0}^{n-1} \times (0, \delta]) \sqcup L^{n-1} \rightarrow (L_{\Theta_0}^{n-1} \times \{\delta\}) \times [0, 1].$$

By the action of the involution  $\tau$ , we get the symmetric homeomorphism  $h_- : L_{\Theta_0}^{n-1} \times \{-\delta\} \rightarrow L^{n-1} = \partial V^n$  and a symmetric respective homeomorphism of cylinders

$$\bar{h}_- : f_-(L_{\Theta_0}^{n-1} \times [-\delta, 0)) \sqcup L^{n-1} \rightarrow (L_{\Theta_0}^{n-1} \times \{-\delta\}) \times [-1, 0].$$

Now it is easy to see that we get the following decomposition of the manifold  $\hat{V}^n$ :

$$\hat{V}^n = (\text{int}(V_+^n) \setminus f_+(L_{\Theta_0}^{n-1} \times (0, \delta))) \bigcup (L_{\Theta_0}^{n-1} \times [-1, 1]) \bigcup (\text{int}(V_-^n) \setminus f_-(L_{\Theta_0}^{n-1} \times (-\delta, 0))).$$

On the left part of this decomposition there is the initial smooth structure  $\Sigma_+$ , on the right part — the initial smooth structure  $\Sigma_-$ , and in the middle there is the smooth structure of the cylinder. Moreover, all these structures are compatible on the boundaries  $L_{\Theta_0}^{n-1} \times \{\pm 1\}$ . Therefore, we get a smooth structure  $\hat{\Sigma}_\varepsilon$  with the needed property.

Lemmas 2 and 1 are completely proved.  $\square$

The proof of the Step 7 is a standard exercise in linear algebra. Therefore, we conclude the proof of Theorem 1.

## 6 Pontrjagin classes and attaching a boundary

For a complex manifold  $M^{2n}$ , which is closed or open and of a finite homotopy type, there is a standard procedure to calculate its Pontrjagin classes from its Chern classes. This calculation for  $\text{Sym}^n M_{g,k}^2$  is not hard and gives the following

**Proposition 1.** *For any  $n \geq 2, g \geq 0, k \geq 1$ , all integral Pontrjagin classes of the manifold  $\text{Sym}^n M_{g,k}^2$  are equal to zero.*

The following fact is not hard to prove and is a folklore.

**Fact  $\theta$ .** *Let us denote by  $D^2$  the closed 2-disk. Then for all  $n \geq 2$  the space  $\text{Sym}^n D^2$  is continuously homeomorphic to the closed 2n-disk  $D^{2n}$ . But, there is no natural smoothing of  $\text{Sym}^n D^2$ .*

**Corollary 1.** *Suppose  $\overline{M}^2$  is a compact 2-manifold with the boundary. Then the space  $\text{Sym}^n \overline{M}^2$  is a compact 2n-manifold with the boundary for all  $n \geq 2$ .*

This corollary implies that we can in TOP category naturally attach a boundary to the open manifold  $\text{Sym}^n M_{g,k}^2$ . We just need to take the compact Riemann surface  $\overline{M}_{g,k}^2$  with  $k$  small open disjointed disks removed (the boundary of these disks can be taken of  $C^\omega$  class). Then for all  $n \geq 2$  we get the topological compact 2n-manifold  $\text{Sym}^n \overline{M}_{g,k}^2$  such that its interior is just  $\text{Sym}^n(\text{int} \overline{M}_{g,k}^2)$ .

Therefore, the interior  $\text{int}(\text{Sym}^n \overline{M}_{g,k}^2)$  has a natural structure of a complex manifold. But, there is no natural smooth structure on the whole  $\partial$ -manifold  $\text{Sym}^n \overline{M}_{g,k}^2$ . Moreover, the following question naturally arises:

**Question 1.** *Could the TOP  $\partial$ -manifold  $\text{Sym}^n \overline{M}_{g,k}^2$  be smoothable?*

It is obvious that the compact 2-manifold  $\overline{M}_{g,k}^2$  possesses a triangulation. For any compact polyhedron  $K$  the space  $\text{Sym}^n K$ ,  $n \geq 2$ , inherits some natural triangulation. So, the  $\partial$ -manifold  $\text{Sym}^n \overline{M}_{g,k}^2$  is a compact polyhedron. But, in dimension 4 the well known fact states that a TOP 4-manifold  $L^4$  (with or without a boundary) is smoothable, if it is a compact polyhedron. Thus, for  $n = 2$  the above Question 1 has a positive answer.

**Proposition 2.** *For  $n \geq 3$  the boundary  $\partial \text{Sym}^n \overline{M}_{g,k}^2$  is a connected closed manifold with a free abelian fundamental group.*

**Proof.** Let us fix any  $n \geq 3, g \geq 0, k \geq 1$ . It is easy to show that the boundary  $\partial \text{Sym}^n \overline{M}_{g,k}^2$  is a connected closed manifold with an abelian fundamental group. The compact  $\partial$ -manifold  $\text{Sym}^n \overline{M}_{g,k}^2$  is orientable, so we can use the standard Poincaré Duality.

Let us use the following notation:  $L^{2n} := \text{Sym}^n \overline{M}_{g,k}^2$ ,  $K^{2n-1} := \partial \text{Sym}^n \overline{M}_{g,k}^2$  and  $\tilde{M}^{2n} := L^{2n} / K^{2n-1}$ .

The Poincaré Duality with  $\mathbb{Z}$ -coefficients gives the commutative diagram:

$$\begin{array}{ccccccc} H^{2n-2}(L^{2n}) & \longrightarrow & H^{2n-2}(K^{2n-1}) & \longrightarrow & H^{2n-1}(\tilde{M}^{2n}) & \longrightarrow & H^{2n-1}(L^{2n}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_2(\tilde{M}^{2n}) & \longrightarrow & H_1(K^{2n-1}) & \longrightarrow & H_1(L^{2n}) & \longrightarrow & H_1(\tilde{M}^{2n}). \end{array}$$

All vertical arrows are isomorphisms and rows are exact.

We know that  $L^{2n} \sim \text{Sk}^n T^s$ ,  $s = 2g + k - 1$ . So, for the dimension reasons we get  $H^{2n-2}(L^{2n}) = 0$ , and  $H^{2n-1}(L^{2n}) = 0$ . From the above diagram we also have  $H_2(\tilde{M}^{2n}) = H_1(\tilde{M}^{2n}) = 0$ . Therefore, the inclusion  $j: K^{2n-1} \hookrightarrow L^{2n}$  induces the isomorphism  $j_*: H_1(K^{2n-1}) \cong H_1(L^{2n}) = \mathbb{Z}^s$ . Above we mentioned that the fundamental group  $\pi_1(K^{2n-1})$  is abelian. So, we get  $\pi_1(K^{2n-1}) = H_1(K^{2n-1}) = \mathbb{Z}^s$ . The proposition 2 is completely proved.  $\square$

Now we can answer the above Question 1 for any  $n \geq 3$ .

**Proposition 3.** *The topological  $\partial$ -manifold  $\text{Sym}^n \overline{M}_{g,k}^2$ ,  $n \geq 3$ , is smoothable. Let us denote by  $\Sigma$  the natural smooth structure on the interior  $\text{int}(\text{Sym}^n \overline{M}_{g,k}^2)$ . Then for any  $\varepsilon > 0$  there exists a smooth structure  $\hat{\Sigma}_\varepsilon$  on the whole manifold  $\text{Sym}^n \overline{M}_{g,k}^2$  such that this structure coincides with the structure  $\Sigma$  on the open subset  $U_\varepsilon := \{x \in \text{Sym}^n \overline{M}_{g,k}^2 \mid d(x, \partial \text{Sym}^n \overline{M}_{g,k}^2) > \varepsilon\}$ . Here  $d(\cdot, \cdot)$  is an arbitrary fixed metric on the metrizable compact  $\text{Sym}^n \overline{M}_{g,k}^2$ .*

To prove this Proposition one just has to take Proposition 2 and an evident version of Lemma 2 above.  $\square$

Let us fix any  $n \geq 2$ ,  $g, g' \geq 0$ ,  $k, k' \geq 1$ , such that  $2g + k = 2g' + k'$  and  $g \neq g'$ . At the end of this paper we want to pose the following

**Conjecture 3.** *The smoothable topological manifolds  $\partial \text{Sym}^n \overline{M}_{g,k}^2$  and  $\partial \text{Sym}^n \overline{M}_{g',k'}^2$  are not homotopy equivalent.*

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